On endoscopy for covering groups of SL(2).

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Plan

- Brief review of Labesse–Langlands (endoscopy for SL_2).
- Endoscopy for 2-fold covering of SL_2.
  - Fundamental lemma for $F/\mathbb{Q}_2$.
  - Generalization of Kohnen's plus space.
- Endoscopy for n-fold covering of SL_2 where n is odd.
  - Endoscopy for $\widetilde{SL_2}$.
  - Covering groups of $D^\times$ and D^1.

Brief Review of Labesse–Langlands

F: p-adic field
$G = SL_2(F)$:

Representation :	L-packet $\Pi_\phi(G)$	Irreducible rep.
Conjugacy class :	Stable conjugacy class	conjugacy class

ϕ: L-parameter

	G		Endoscopic group H
	invariant distribution	$\xleftarrow{Tran_H^G}$	stable distribution
Rep.	$\sum_{\pi \in \Pi_\phi(G)} c_\pi J(\pi)$	$\longleftarrow\!\!\shortmid$	$J^{st}(\phi^H)$
Conj.	$\sum_{g \in \mathcal{O}^{st}} \Delta(h,g) J(g)$	$\longleftarrow\!\!\shortmid$	$J^{st}(h)$

h: regular semisimple

Elliptic endoscopic group H of G are one of the following type.

G

$E^1 = \{x \in E^\times \mid N_{E/F}(x) = 1\}, \quad E/F$: quadratic extension

We regard H as an subgroup of G.

$$H \subset G$$

$$a + b\tau \in E^1 \mapsto \begin{pmatrix} a & bv \\ b & a + bu \end{pmatrix}$$

$$E = F(\tau)$$
$$\tau^2 = u\tau + v$$

$G_{\text{reg}} = \{g \in G |\, g$ is regular semisimple$\}$

g is regular semisimple if Cent$(g, G) \simeq GL_1$ is a torus.

$H_{G\text{-reg}} = \{h \in H |\, h$ is regular semisimple in $G\}$

We say that $g, g' \in G$ is "stably conjugate" if there exists $x \in GL_2(F)$ such that.

$$g' = x^{-1} g x$$

We say that $h \in H_{G\text{-reg}}$ is an "image" of $g \in G_{\text{reg}}$ if h is stably conjugate to g in G.

$$\{\text{stable conjugacy class of } G_{\text{reg}}\} \longleftarrow \{\text{stable conjugacy class of } H_{G\text{-reg}}\}$$

We can define a transfer factor

$$\Delta_{G,H}(h, g) \in \mathbb{C}^{\times}, \quad h \in H_{G\text{-reg}} \text{ is an image of } g \in G_{\text{reg}}.$$

$$\sum_{g \in G_{\text{reg}}/\sim} \Delta(h,g) J(g,\,\cdot\,) \longleftarrow J^{st}(h,\,\cdot\,)$$

We say that $f \in C_c^\infty(G)$ and $f^H \in C_c^\infty(H)$ have "matching orbital integrals" if

$$\sum_{g \in G_{\text{reg}}/\sim} \Delta_{G,H}(h,g) J(g,f) = J^{st}(h,f^H), \quad {}^\forall h \in H_{G\text{-reg}}$$

Existence of transfer

For each $f \in C_c^\infty(G)$ there exists $f^H \in C_c^\infty(H)$ such that f and f^H have matching orbital integrals.

$J(g,f) = D(g) \int_{\text{Cent}(g,G)\backslash G} f(x^{-1}gx)\,dx$
$D(g)$: Weyl denominator
$G_{\text{reg}}/\sim$: set of conjugacy classes in G_{reg}
$\Delta_{G,H}(h,g) = 0$ if h is not an image of g
$J^{st}(h,f^H) = \sum_{h' \sim_{st} h} J(h',f^H)$

$$\{\text{distribution on } G\} \overset{\mathrm{Tran}_H^G}{\longleftarrow} \{\text{stable distribution on } H\}$$

For stable distribution S on H, we define an invariant distribution $\mathrm{Tran}_H^G S$ by

$$\mathrm{Tran}_H^G S(f) = S(f^H),$$

where f and f^H have matching orbital integrals.

We say that S is "stable" if

$$S(f^H) = 0, \quad {}^\forall f^H \in C_c^{\infty,-}(H),$$

where

$$C_c^{\infty,-}(H) = \{f^H \in C_c^\infty(H) |\, J^{st}(h, f^H) = 0, \quad {}^\forall h \in H_{\mathrm{reg}}\}$$

For $H = SL_2$ or E^1 for unramified E/F we have a homomorphism

$$\lambda : \mathcal{H}(G, K) \longrightarrow \mathcal{H}(H, K^H)$$

$\mathcal{H}(G, K)$: Hecke algebra

$K = SL_2(\mathfrak{o}_F)$

$K^H = H(\mathfrak{o}_F)$

$\mathfrak{o}_F$: ring of integer of F.

Fundamental lemma

For any $f \in \mathcal{H}(G, K)$, functions f and $\lambda(f)$ have matching orbital integrals.

Packet and endoscopy

$$\sharp\Pi_\phi(G) = 1, 2, 4$$

Endoscopic groups for $\Pi_\phi(G)$ are

$$\begin{cases} G, & \sharp\Pi_\phi(G) = 1 \\ G, \text{ one of } E^1, & \sharp\Pi_\phi(G) = 2 \\ G, E^1_1, E^1_2, E^1_3, & \sharp\Pi_\phi(G) = 3 \end{cases}$$

Case $H = G$

$$\sum_{\pi \in \Pi_\phi(G)} J(\pi) = \mathrm{Tran}_H^G(\text{stable distribution})$$

is stable.

$J(\pi)$: distribution character of π.

Packet and endoscopy

If $\sharp\Pi_\phi(G) = 2$, then there exists a character π^H of $H = E^1$ such that

$$J(\pi_1) - J(\pi_2) = \mathrm{Tran}_H^G J(\pi^H)$$

If $\sharp\Pi_\phi(G) = 4$, then there exists a character π_i^H of E_i^1 $(i = 1, 2, 3)$ such that

$$J(\pi_1) + J(\pi_2) - J(\pi_3) - J(\pi_4) = \mathrm{Tran}_{E_1^1}^G \pi_1^H$$
$$J(\pi_1) - J(\pi_2) + J(\pi_3) - J(\pi_4) = \mathrm{Tran}_{E_2^1}^G \pi_2^H$$
$$J(\pi_1) - J(\pi_2) - J(\pi_3) + J(\pi_4) = \mathrm{Tran}_{E_3^1}^G \pi_3^H$$

if you number $\pi_1, \ldots, \pi_4$ properly.

Correspondence between stable conjugacy of G and H

Existence of transfer

Fundamental lemma

$$\Downarrow$$

Lift from representation of H to G

Description of the packets

(We use twisted endoscopy and twisted trace formula of $GL_2(F)$ to get the results.)

n-fold covering group of $G = SL_2(F)$

F: p-adic field.

$\mathfrak{o}_F$: ring of integers of F.

$\mathfrak{p}_F$: prime ideal in $\mathfrak{o}_F$.

μ_n: n-th roots of 1 in $F^\times$.

Assumption

We assume

$$\sharp\mu_n = n,$$

i.e., all n-th roots of 1 are contained in $F^\times$.

We can define n-th power norm residue symbol.

Definition

n-fold covering group $\widetilde{G}$

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

is defined by

$$[g_1, \zeta_1][g_2, \zeta_2] = [g_1 g_2\,,\, \zeta_1 \zeta_2 \mathbf{c}(g_1, g_2)],$$
$$g_1, g_2 \in G,\ \zeta_1, \zeta_2 \in \mu_n$$

where $\mathbf{c}$ is Kubota's 2-cocycle:

$$\mathbf{c}(g_1, g_2) = \left\langle \frac{\mathbf{x}(g_1)}{\mathbf{x}(g_1 g_2)}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1 g_2)} \right\rangle_n$$

$$\mathbf{x}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c, & c \neq 0 \\ d, & c = 0 \end{cases}$$

$\langle\ ,\ \rangle_n$: n-th power norm residue symbol

Let $C_c^\infty(\widetilde{G})$ be the space of locally constant compactly supported function $\widetilde{f}$ on $\widetilde{G}$ such that

$$\widetilde{f}(\widetilde{g} \cdot [1, \zeta]) = \zeta^{-1} \widetilde{f}(\widetilde{g}), \quad {}^\forall \widetilde{g} \in \widetilde{G}, \zeta \in \mu_n$$

(Anti-genuine function.)

For subset A in G, we denote by $\widetilde{A}$ the inverse image of A in $\widetilde{G}$. For example,

$$\widetilde{G_{\text{reg}}} = \{[g, \zeta] | \, g \in G_{\text{reg}}, \, \zeta \in \mu_n\}$$
$$\widetilde{SL_2(\mathfrak{o}_F)} = \{[k, \zeta] | \, k \in SL_2(\mathfrak{o}_F), \, \zeta \in \mu_n\}$$

If $(n, p) = 1$ then we have a splitting

$$\mathbf{s} : SL_2(\mathfrak{o}_F) \longrightarrow \widetilde{SL_2(\mathfrak{o}_F)}$$

Definition

For $\tilde{g} \in \widetilde{G_{\text{reg}}}$, we say that $\tilde{g}$ is "relevant" if

$$\tilde{x}^{-1}\tilde{g}\tilde{x} = \tilde{g}$$

for any $\tilde{x} \in \widetilde{\text{Cent}(g, G)}$. We put

$$\tilde{G}_{\text{rel}} = \{\tilde{g} \in \tilde{G} | \, \tilde{g} \text{ is relevant.}\}$$

If $\tilde{g}$ is not relevant then $^{\exists}\tilde{x} \in \widetilde{\text{Cent}(g, G)}$ s.t.

$$\tilde{x}^{-1}\tilde{g}\tilde{x} = \tilde{g}[1, \zeta'], \quad \zeta' \neq 1.$$

Hence

$$J(\tilde{g}, \tilde{f}) = 0.$$

$$J(\tilde{g}, \tilde{f}) = D(g) \int_{\text{Cent}(\tilde{g}, \tilde{G}) \backslash \tilde{G}} \tilde{f}(\tilde{x}^{-1}\tilde{g}\tilde{x}) \, d\tilde{x}$$

$$n_0 = \begin{cases} n/2, & n : \text{even} \\ n, & n : \text{odd} \end{cases}$$

$\tilde{g} = [g, \zeta] \in \widetilde{G_{\mathrm{reg}}}$ is relevant if and only if there exists $h \in G_{\mathrm{reg}}$ s.t.

$$g = h^{n_0} \text{ or } -h^{n_0}$$

Elliptic endoscopic group of $\widetilde{G}$

n : even	PGL_2, PGL_2
n : odd	SL_2, E^1 (E/F: quadratic ext.)

- J. P. Schultz: $n = 2$
- A. Trehan: $\widetilde{SL_N}$ $(n|N)$
- Wen–Wei Li: 2-fold covering group of $Sp(2N)$

$n = 2$

We put

$$H^+ = H^- = PGL_2(F)$$

We define $\tau^+, \tau^- : PGL_2(F) \longrightarrow SL_2(F)$ by

$$\begin{array}{ccc} GL_2(F) \ni h & \longrightarrow & SL_2(F) \ni \det(h)^{-1}h^2 \\ \downarrow & \nearrow_{\tau^+} & \\ H^+ = PGL_2(F) & & \end{array}$$

$$\begin{array}{ccc} GL_2(F) \ni h & \longrightarrow & SL_2(F) \ni -\det(h)^{-1}h^2 \\ \downarrow & \nearrow_{\tau^-} & \\ H^- = PGL_2(F) & & \end{array}$$

We say that $h \in H^{\pm}(F)$ is an image of $[g, \zeta] \in \widetilde{G}$ if

$$\tau^{\pm}(h) \sim_{st} g.$$

$\sim_{st}$: stably conjugate.

Definition

If $h \in H^{+}(F)$ is an image of $[g,\zeta] \in \widetilde{G}$ then we define a transfer factor Δ_{ψ}^{+} by

$$\Delta_{\psi}^{+}(h,[g,\zeta]) = \zeta \frac{\alpha_{\psi}(1)}{\alpha_{\psi}(\det h)} \mathbf{c}(\det h \cdot 1_2, g)$$

If $h \in H^{-}(F)$ is an image of $[g,\zeta] \in \widetilde{G}$ then we define a transfer factor Δ_{ψ}^{-} by

$$\Delta_{\psi}^{-}(h,[g,\zeta]) = \alpha_{\psi}(1)^2 \mathbf{c}(-1_2, g) \Delta_{\psi}^{+}(h\,,\,[-1,1][g,\zeta])$$

ψ: non-trivial additive character of F

$\gamma_{\psi}(x)$: Weil constant

$$\int_F \phi(t)\psi(xt^2)\,dt = \alpha_{\psi}(x)|x|^{-1/2} \int_F \widehat{\phi}(t)\psi(-t^2/4x)\,dt, \qquad \phi \in \mathcal{S}(F)$$

where

$$\widehat{\phi}(t) = \int_F \phi(u)\psi(tu)\,du_{\psi}$$

is the Fourier transform of ϕ and du_{ψ} is the self-dual Haar measure.

Theorem

"Existence of transfer" holds.

Theorem

If p is odd then "Fundamental Lemma" holds.

Theorem (Schultz)

Let π be an irreducible admissible representation of $PGL_2(F)$. Then there are two admissible representations $\widetilde{\pi}^+$ and $\widetilde{\pi}^-$ of $\widetilde{G}$, which are either irreducible or zero, such that

$$\mathrm{Tran}_{H^+}^{\widetilde{G}}(J(\pi)) = J(\widetilde{\pi}^+) + J(\widetilde{\pi}^-)$$
$$\mathrm{Tran}_{H^-}^{\widetilde{G}}(J(\pi)) = J(\widetilde{\pi}^+) - J(\widetilde{\pi}^-)$$

$\widetilde{\pi}^+, \widetilde{\pi}^-$ are described by theta correspondence and ϵ-factor.

Fundamental lemma for $F/\mathbb{Q}_2$

ψ: non-trivial additive character of F
c_ψ: maximum integer c such that $\psi(\mathfrak{p}_F^c) = 1$
$\mathfrak{c} = \mathfrak{p}^{c_\psi}$

ω_ψ: Weil representation acting on $\mathcal{S}(F)$

$(\phi_1, \phi_2) = \int_F \phi_1(x)\overline{\phi_2(x)}\, dx, \quad \phi_1, \phi_2 \in \mathcal{S}(F)$

Haar meausure on F is normalized so that $\mathrm{Vol}(\mathfrak{o}_F) = 1$.

For ideals $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{o}_F$ we put

$$\Gamma[\mathfrak{a}, \mathfrak{b}] = \left\{ \begin{pmatrix} \mathfrak{o}_F & \mathfrak{a} \\ \mathfrak{b} & \mathfrak{o}_F \end{pmatrix} \right\} \cap G$$

We put

$$\Gamma = \Gamma[\mathfrak{c}^{-1}, 4\mathfrak{c}]$$

We define a (anti-) genuine character $\epsilon : \widetilde{\Gamma} \longrightarrow \mathbb{C}^\times$ by

$$\omega_\psi(g)\phi_0 = \epsilon(g)^{-1}\phi_0,$$

where $\phi_0 \in \mathcal{S}(F)$ is the characteristic function of $\mathfrak{o}_F$.

The Hecke algebra $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(\widetilde{SL_2(F)}, \widetilde{\Gamma}; \epsilon)$ is the space of (anti-) genuine function $\widetilde{\varphi} \in \widetilde{C}_c^\infty(\widetilde{G})$ such that

$$\widetilde{\varphi}(\widetilde{\gamma}_1\widetilde{g}\widetilde{\gamma}_2) = \epsilon(\widetilde{\gamma}_1)\epsilon(\widetilde{\gamma}_2)\widetilde{\varphi}(\widetilde{g}), \quad \widetilde{\gamma}_1, \widetilde{\gamma}_2 \in \widetilde{\Gamma}.$$

We define an idempotent $e^K \in \widetilde{\mathcal{H}}$ by

$$e^K(\widetilde{g}) = \begin{cases} |2|_F^{-1}(\phi_0, \omega_\psi(\widetilde{g})\phi_0), & \widetilde{g} \in \widetilde{\Gamma[\mathfrak{c}^{-1}, \mathfrak{c}]} \\ 0, & \text{otherwise} \end{cases}$$

$$\widetilde{\mathcal{H}}^{e^K} = e^K * \widetilde{\mathcal{H}} * e^K \longrightarrow \mathcal{H} = \mathcal{H}(PGL_2(F), PGL_2(\mathfrak{o}_F))$$

Fundamental lemma

"Fundamental lemma" holds for H^+ and H^-.

We put

$$E^K(\widetilde{g}) = e^K(\mathbf{w}_2^{-1}\widetilde{g}\mathbf{w}_2),$$

where

$$\mathbf{w}_2 = \left[\begin{pmatrix} 0 & -2^{-1} \\ 2 & 0 \end{pmatrix}, 1\right]$$

For general $F/\mathbb{Q}_p$, we define e^K and E^K similarly.

Kohnen plus space

$F = \mathbb{Q}$

$\mathfrak{h} = \{z \in \mathbb{C} | \, \mathrm{Im} z > 0\}$

$S_{\kappa+(1/2)}(\Gamma_0(4))$: space of cusp forms

$$j^{\kappa+(1/2)}(\gamma, z) = (j^{1/2}(\gamma, z))^{2\kappa+1}, \ \gamma \in \Gamma_0(4), \ z \in \mathfrak{h}$$

$$\theta(\gamma(z)) = j^{1/2}(\gamma, z)\theta(z)$$

$$\theta(z) = \sum_{x \in \mathbb{Z}} \exp(2\pi\sqrt{-1}x^2 z)$$

Definition (Kohnen plus space)

$S_{\kappa+(1/2)}(\Gamma_0(4))$ is the space of $h \in S_{\kappa+(1/2)}(\Gamma_0(4))$ with Fourier expansion of the form

$$h(z) = \sum_{(-1)^\kappa N \equiv 0,1 \mod 4} c(N) \exp(2\pi\sqrt{-1}Nz)$$

Kohnen

As a Hecke module $S^+_{\kappa+(1/2)}(\Gamma_0(4))$ is isomorphic to $S_{2\kappa}(SL_2(\mathbb{Z}))$.

Generalization of Kohnen plus space

$\mathbf{F}$: totally real number field of degree l over $\mathbb{Q}$

$\mathbb{A}$: ring of adele of $\mathbf{F}$

$\mathfrak{o}_{\mathbf{F}}$: integer ring of $\mathbf{F}$

$\mathfrak{d}_{\mathbf{F}}$: different for $\mathbf{F}/\mathbb{Q}$

For

$$\kappa = (\kappa_1, \ldots, \kappa_l) \in \mathbb{Z}^l_{\geq 0}$$

we fix a unit $\eta \in \mathfrak{o}_{\mathbf{F}}^{\times}$ such that

$$N_{\mathbf{F}/\mathbb{Q}}(\eta) = \prod_{i=1}^{l} (-1)^{\kappa_i}.$$

Let ψ be the additive character of $\mathbb{A}/\mathbf{F}$ such that

$$\psi_v(x) = \exp(2\pi\sqrt{-1}\eta_v x), \quad {}^{\forall}v: \text{ real.}$$

We put

$$E^K = \prod_{v<\infty} E_v^K$$

$$\mathcal{A}^{cusp}_{\kappa+(1/2)}(SL_2(\mathbf{F})\backslash \widetilde{SL_2(\mathbb{A})})^{E^K}$$
$$= \{\phi \in \mathcal{A}^{cusp}_{\kappa+(1/2)}(SL_2(\mathbf{F})\backslash \widetilde{SL_2(\mathbb{A})}) \mid \rho(E^K)\phi = \phi\},$$

where ρ is the right regular representation of $\widetilde{SL_2(\mathbb{A})}$.

Theorem

Assume $\kappa_i > 1$ for some $i = 1, \ldots, l$, then

$$\mathcal{A}^{cusp}_{\kappa+(1/2)}(SL_2(\mathbf{F})\backslash \widetilde{SL_2(\mathbb{A})})^{E^K} \overset{1:1}{\longleftrightarrow} \mathcal{A}^{cusp}_{2\kappa}(PGL_2(\mathbf{F})\backslash PGL_2(\mathbb{A})/\mathcal{K}_0)$$

$$\mathcal{K}_0 = \textstyle\prod_{v<\infty} PGL_2(\mathfrak{o}_{\mathbf{F}_v})$$

Let

$$S_{\kappa+(1/2)}(\Gamma)^{E^K} \subset S_{\kappa+(1/2)}(\Gamma)$$

be the subspace corresponding to the subspace

$$\mathcal{A}^{cusp}_{\kappa+(1/2)}(SL_2(\mathbf{F})\backslash \widetilde{SL_2(\mathbb{A})})^{E^K}.$$

The factor of automorphy is given by

$$j^{\kappa+(1/2),\eta}(\gamma, z) = \prod_{v<\infty} \epsilon_v([\gamma, 1]) \prod_{i=1}^{l} \tilde{j}([\iota_i(\gamma), 1], z_i)^{2\kappa_i+1}.$$

$$\Gamma = \Gamma[\mathfrak{d}_{\mathbf{F}}^{-1}, 4\mathfrak{d}_{\mathbf{F}}]$$

$$\tilde{j}\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta\right], z\right) = \begin{cases} \zeta\sqrt{d}, & c = 0,\ d > 0 \\ -\zeta\sqrt{d}, & c = 0,\ d < 0 \\ \zeta(cz+d)^{1/2}, & c \neq 0 \end{cases}$$

$\iota_1, \ldots, \iota_l$ are embeddings of F into $\mathbb{R}$.

Theorem

Assume $\kappa_i > 1$ for some $i = 1, \ldots, l$, then

$$\mathcal{A}^{cusp}_{\kappa+(1/2)}(SL_2(\mathbf{F})\backslash \widetilde{SL_2(\mathbb{A})}; \epsilon)^{E^K} \overset{1:1}{\longleftrightarrow} \mathcal{A}^{cusp}_{2\kappa}(PGL_2(\mathbf{F})\backslash PGL_2(\mathbb{A})/\mathcal{K}_0)$$

Moreover

$$S_{\kappa+(1/2)}(\Gamma)^{E^K} = S^+_{\kappa+(1/2)}(\Gamma),$$

where $S^+_{\kappa+(1/2)}(\Gamma)$ is the space of $h(z) \in S^+_{\kappa+(1/2)}(\Gamma)$ with Fourier expansion of the form

$$h(z) = \sum_{\xi \equiv \square \mod 4} c(\xi) \exp(2\pi\sqrt{-1}\xi z)$$

$\xi \in \mathfrak{o}_F$

"$\xi \equiv \square \mod 4$" means $^\exists x \in \mathfrak{o}_F$ s.t. $\xi \equiv x^2 \mod 4$

(M. Ueda studied the relation for higher level Γ.)

Theorem

For $h \in S_{\kappa+(1/2)}(\Gamma)^{E^K} \cap \mathcal{A}_{00}$ and totally positive $\xi \in \mathbf{F}^\times$ we have

$$\frac{|c_\xi|^2}{\langle \widetilde{\varphi_h}, \widetilde{\varphi_h} \rangle} = \mathcal{D}_K^{1/2} 2^{-1+3|\kappa|} \xi_F(2) \frac{L(1/2, \tau \otimes \widehat{\chi}_{\eta\xi})}{L(1, \tau, Ad)}$$

$h \longleftrightarrow \widetilde{\varphi_h} \in \mathcal{A}_{\kappa+(1/2)}(SL_2(\mathbf{F}) \backslash \widetilde{SL_2(\mathbb{A})})^{E^K}$

$\widetilde{\varphi_h} \longleftrightarrow \tilde{\sigma}$ automorphic representation of $\widetilde{SL_2}$

τ: automorphic representation of PGL_2 corresponding to $\tilde{\sigma}$

$\mathcal{A}_{00}$: The space of cusp forms orthogonal to the Weil representations associated to one-dimensional quadratic forms.

$\widehat{\chi}_a$: Hecke character of $\mathbb{A}^\times$ corr. to $F(\sqrt{a})/F$.

c_ξ: given by the ξ-th Fourier coefficient of h.

$\langle \widetilde{\varphi_h}\,, \widetilde{\varphi_h} \rangle = \int_{SL_2(\mathbf{F}) \backslash SL_2(\mathbb{A})} |\widetilde{\varphi_h}(x)|\, dx$

$\mathcal{D}_F$: discriminant of F

ξ_F: complete Dedekind zeta

Case n is odd

n: odd

We have a strict correspondence between $SL_2(F)$ and $\widetilde{SL_2(F)}$.

Definition

We say that $h \in SL_2(F)_{\mathrm{reg}}$ is a strict image of $[g, \zeta] \in \widetilde{SL_2(F)}_{\mathrm{rel}}$ if

$$h^n \sim g,$$

where $\sim$ means the conjugacy in $SL_2(F)$.

Definition (Transfer factor)

We define the transfer factor Δ_{st} by

$$\Delta_{st}(h, [g, \zeta]) = \begin{cases} \zeta, & h \text{ is a strict image of } [g, \zeta] \\ 0, & \text{otherwise} \end{cases}$$

For the above correspondence, we have

Theorem

"Existence of transfer" holds.

If $(p, n) = 1$ then "Fundamental lemma" holds.

Endoscopy for $\widetilde{SL_2(F)}$

Elliptic endoscopic group of $\widetilde{SL_2(F)}$

$$H = SL_2(F) \text{ or } E^1$$

$$H \rightsquigarrow SL_2(F) \rightsquigarrow \widetilde{SL_2(F)}$$

Theorem

"Existence of transfer" holds for H.

If $(p, n) = 1$ then "Fundamental lemma" holds for H.

Theorem

There exists packets

$$\Pi(\widetilde{G}) = \{\widetilde{\pi}\}$$
$$\text{or } \Pi(\widetilde{G}) = \{\widetilde{\pi}_1, \widetilde{\pi}_2\}$$
$$\text{or } \Pi(\widetilde{G}) = \{\widetilde{\pi}_1, \widetilde{\pi}_2, \widetilde{\pi}_3, \widetilde{\pi}_4\}$$

If $\sharp\Pi(G) = 2$

$$J(\widetilde{\pi}_1) - J(\widetilde{\pi}_2) = c \cdot \mathrm{Tran}_{E^1}^G J(\pi^H)$$

If $\sharp\Pi_\phi(G) = 4$, then there exists a character π_i^H of E_i^1 $(i = 1, 2, 3)$ such that

$$J(\widetilde{\pi}_1) + J(\widetilde{\pi}_2) - J(\widetilde{\pi}_3) - J(\widetilde{\pi}_4) = c_1 \cdot \mathrm{Tran}_{E_1^1}^G \pi_1^H$$
$$J(\widetilde{\pi}_1) - J(\widetilde{\pi}_2) + J(\widetilde{\pi}_3) - J(\widetilde{\pi}_4) = c_2 \cdot \mathrm{Tran}_{E_2^1}^G \pi_2^H$$
$$J(\widetilde{\pi}_1) - J(\widetilde{\pi}_2) - J(\widetilde{\pi}_3) + J(\widetilde{\pi}_4) = c_3 \cdot \mathrm{Tran}_{E_3^1}^G \pi_3^H$$

if you number $\widetilde{\pi}_1, \ldots, \widetilde{\pi}_4$ properly.

Covering of $D^\times$

D: quaternion algebra

D^1: group of reduced norm 1 elements

$$\begin{array}{ccc} \boxed{?} & \qquad & \widetilde{GL_2(F)} \\ \downarrow & & \downarrow \\ D^\times & & GL_2(F) \end{array}$$

$$\begin{array}{ccc} \boxed{?} & \qquad & \widetilde{SL_2(F)} \\ \downarrow & & \downarrow \\ D^1 & & SL_2(F) \end{array}$$

We should have

$$D^{\times}_{\text{rel}}/\!\sim \;\longrightarrow\; GL_2(F)_{\text{rel}}/\!\sim$$

E/F: quadratic extension
$E^{\times} \subset GL_2(F)$

$$[x,1]^{-1}[y,1][x,1] = [y, \langle y, \overline{x}\rangle_{E,n}]$$

In $GL_2(E)$

$$\mathbf{c}\left(\begin{pmatrix} y & 0 \\ 0 & \overline{y} \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix}\right) = \langle y, \overline{x}\rangle_{E,n}$$

Square root of the 2-cocycle?

$$\langle\, ,\,\rangle_{E,n}^{m}, \quad 2m \equiv 1 \mod n$$
$$\langle\, ,\,\rangle_{E,2n}$$

E/F: quadratic extension

Definition

We construct a covering group $\widetilde{D^\times}$ by the following way. Let $\widetilde{GL_2(E)}$ be the covering of $GL_2(E)$ by

$$\mathbf{c}(g_1, g_2) = \left\langle \frac{\mathbf{x}(g_1)}{\mathbf{x}(g_1 g_2)}, \det g_1 \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1 g_2)} \right\rangle_{E,n}^{m}.$$

where $2m \equiv 1 \mod n$.

$$\begin{array}{ccc} \widetilde{D^\times} & \longrightarrow & \widetilde{GL_2(E)} \\ \downarrow & & \downarrow \\ D^\times & \longrightarrow & GL_2(E) \end{array}$$

We define $\widetilde{D^\times}$ by the pull-back of the image of $D^\times \longrightarrow GL_2(E)$.

We can also construct $\widetilde{D^\times}$ from the covering of $GL_2(E)$ defined by the 2-cocycle

$$\mathbf{c}'(g_1, g_2) = \left\langle \frac{\mathbf{x}(g_1)}{\mathbf{x}(g_1 g_2)}, \det g_1 \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1 g_2)} \right\rangle_{E,2n}$$

For a quaternion algebra $\mathbf{D}$ over $\mathbf{F}$, we can construct $\widetilde{\mathbf{D}_{\mathbb{A}}^\times}$ similarly. Then for any place v where $\mathbf{D}_v^1$ splits, the above covering group $\widetilde{\mathbf{D}_v^\times}$ is isomorphic to the usual $\widetilde{GL_2(\mathbf{F}_v)}$. Moreover we have a splitting

$$\mathbf{D}^\times \longrightarrow \widetilde{\mathbf{D}_{\mathbb{A}}^\times}.$$

Similar statements hold for $\widetilde{D^1}$.

Theorem

"Existence of transfer" holds for strict correspondence between $\widetilde{D^\times}$ and $D^\times$.

Theorem

For irreducible rep. π of $D^\times$ there exists $\tilde{\pi}$ such that

$$J(\tilde{\pi}) = c\mathrm{Tran}_{D^\times}^{\widetilde{D^\times}} J(\pi)$$

Theorem?

We have a description of the packets for $\widetilde{D^1}$.

Thank you!